What are these in terms of a, b, and c?

$\sin(\theta) =$ b/c $= \sqrt{3}/2$

$\cos(\theta) =$ a/c $= 1/2$

$\tan(\theta) =$ b/a $= \sqrt{3}/1$

$\sec(\theta) =$ c/a $= 2$

$1^2 + b^2 = 2^2$

$b^2 = 3$
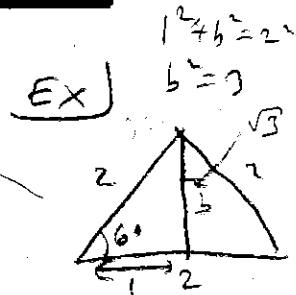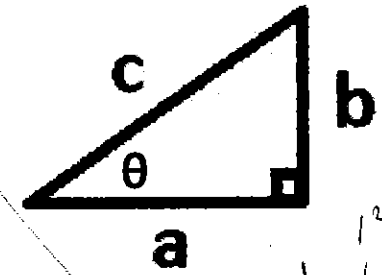
7.2 Trig Integrals (*continued*)

Entry Task: Fill in the blanks

Square Identities
$\sin^2(x) = 1 - \cos^2(x)$
$\cos^2(x) = 1 - \sin^2(x)$
$\tan^2(x) = \sec^2(x) - 1$
$\sec^2(x) = 1 + \tan^2(x)$
Half Angle Identities
$\sin^2(x) = \dfrac{1}{2}\left(1 - \cos(2x)\right)$
$\cos^2(x) = \dfrac{1}{2}\left(1 + \cos(2x)\right)$
$\sin(x)\cos(x) = \dfrac{1}{2}\sin(2x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = 2\cos^2(x) - 1$

Integrate ^{odd}

$$\int \sin^3(x)\cos^2(x)\,dx$$

$\int \sin^2(x)\cos^2(x)\,\sin(x)\,dx$

$\int (1 - \cos^2(x))\cos^2(x)\,\sin(x)\,dx$

$\int (1 - u^2)u^2\,(-1)\,du$

$= \int -u^2 + u^4\,du$

$= -\dfrac{1}{3}u^3 + \dfrac{1}{5}u^5 + c$

$= \boxed{-\dfrac{1}{3}\cos^3(x) + \dfrac{1}{5}\cos^5(x) + c}$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$(-1)du = \sin(x)\,dx$

Integrals involving sin(x) and cos(x)
Case 1 (*cos(x) or sin(x) has odd power*)

- Separate one from the odd power.
  (*i.e.* pull out one sin(x) or cos(x))
- Use $\quad \sin^2(x) = 1 - \cos^2(x)$
  $\qquad \cos^2(x) = 1 - \sin^2(x)$
  (get rest in term of the other)
- Use u-substitution.

$\swarrow$ BOTH ODD
$\qquad\qquad L$

$$\int \sin^3(x)\cos^5(x)\,dx$$

$\int \sin^2(x)\cos^5(x)\sin(x)\,dx$

$\int (1-\cos^2(x))\cos^5(x)\sin(x)\,dx \qquad u = \cos(x)$

$\qquad\qquad\qquad\qquad\qquad\qquad du = -\sin(x)dx$

$\int (1-u^2)u^5\,(-1)\,du$

$\int -u^5 + u^7\,du = -\frac{1}{6}u^6 + \frac{1}{8}u^8 + c$

$\qquad\qquad = -\frac{1}{6}\cos^6(x) + \frac{1}{8}\cos^8(x) + c$

Integrals involving sin(x) and cos(x)

Case 1 (*cos(x) or sin(x) has odd power*)

- Separate one from the odd power. (*i.e.* pull out one sin(x) or cos(x))
- Use $\sin^2(x) = 1 - \cos^2(x)$
  $\cos^2(x) = 1 - \sin^2(x)$
  (get rest in term of the other)
- Use u-substitution.

Case 2 (*both sin(x),cos(x) even powers*)

- Use

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

over and over again until you can integrate.

Example:

$$\int \cos^4(x)\, dx$$

$$= \int \cos^2(x) \cos^2(x)\, dx$$

$$= \int \frac{1}{2}(1 + \cos(2x)) \frac{1}{2}(1 + \cos(2x))\, dx$$

$$= \frac{1}{4} \int 1 + 2\cos(2x) + \cos^2(2x)\, dx$$

$$= \frac{1}{4} \int 1 + 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\, dx$$

$$= \frac{1}{4} \int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\, dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right] + C$$

$$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$$

Integrals involving tan(x) and sec(x)

Case 3 (sec(x) has an even power)

- Separate out $\sec^2(x)$
- Use $\sec^2(x) = 1 + \tan^2(x)$
  (get rest in terms of $\tan(x)$)
- Use $u = \tan(x)$

Example: $\int \tan^5(x)\sec^4(x)dx$

$$\int \tan^5(x) \sec^2(x) \sec^2(x)dx$$

$$\int \tan^5(x)\left(1+\tan^2(x)\right)\sec^2(x)dx$$

$$\int u^5(1+u^2)du \qquad \begin{array}{l} u = \tan(x)dx \\ du = \sec^2(x)dx \end{array}$$

$$\int u^5 + u^7 du$$

$$= \frac{1}{6}u^6 + \frac{1}{8}u^8 + c$$

$$= \boxed{\frac{1}{6}\tan^6(x) + \frac{1}{8}\tan^8(x) + c}$$

Case 4 (tan(x) has an odd power) AND AT LEAST ONE sec(x)

- Separate out $\sec(x)\tan(x)$
- Use $\tan^2(x) = \sec^2(x) - 1$
  (get rest in terms of $\sec(x)$)
- Use $u = \sec(x)$

Example: $\int \tan^3(x)\sec(x)dx$

$$\int \tan^2(x)\, \sec(x)\, \tan(x)\, dx$$

$$\int (\sec^2(x) - 1)\, \sec(x)\, \tan(x)\, dx$$

$$u = \sec(x)$$
$$du = \sec(x)\tan(x)\, dx$$

$$\int u^2 - 1\, du$$

$$\frac{1}{3}u^3 - u + c$$

$$\boxed{\frac{1}{3}\sec^3(x) - \sec(x) + c}$$

And if you've tried the four cases and are
stuck, here are things to try:
1. Rewrite in terms of sin(x) and cos(x).
2. Rewrite in terms of sec(x) and tan(x).
3. Try using trig identities.

There are still a few "holes".
Namely, when there is one tan(x) or an
odd power on sec(x). You can quote
these (proof in the book):

$Ex)$ $\int \dfrac{\cos(x) + \sin(2x)}{\sin(x)} dx$ to start?

$= \int \dfrac{\cos(x) + 2\sin(x)\cos(x)}{\sin(x)} dx$

— — — — — — —

$\int \dfrac{\tan(x)\cos(x)}{\sec(x)} dx = \int \dfrac{\frac{\sin(x)}{\cos(x)}\cos(x)}{(\frac{1}{\cos(x)})} dx$

$= \int \sin(x)\cos(x) dx$

$$\int \tan(x)\, dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x)\, dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C$$

7.3 Trigonometric Substitution

Goal: Develop a method to evaluate integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$

CASE	SUBSTITUTION
$a^2 - x^2$	$x = a\sin(\theta)$, $-\pi/2 \le \theta \le \pi/2$
$a^2 + x^2$	$x = a\tan(\theta)$, $-\pi/2 < \theta < \pi/2$
$x^2 - a^2$	$x = a\sec(\theta)$, $0 \le \theta < \pi/2$, $\pi \le \theta < 3\pi/2$

Example:

1. $\displaystyle \int \frac{x^3}{\sqrt{4 - x^2}}\,dx$

$\boxed{x = 2\sin\theta}$
$dx = 2\cos\theta\,d\theta$

$= \displaystyle\int \frac{8\sin^3\theta}{\sqrt{4 - 4\sin^2\theta}}\, 2\cos\theta\,d\theta$

$= \displaystyle\int \frac{8\sin^3\theta}{2\cos\theta}\, 2\cos\theta\,d\theta$

$= 8\displaystyle\int \sin^3\theta\,d\theta$

KEY NOTE

$4 - 4\sin^2\theta = 4(1 - \sin^2\theta)$
$\qquad = 4\cos^2\theta$
$\Rightarrow \sqrt{4 - 4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$8\displaystyle\int \sin^2\theta\,\sin\theta\,d\theta$

$8\displaystyle\int (1 - \cos^2\theta)\sin\theta\,d\theta$

$8\displaystyle\int (1 - u^2)(-1)\,du$

$\quad u = \cos\theta$
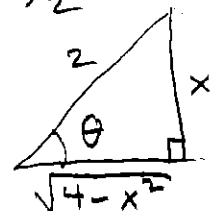$\quad du = -\sin\theta\,d\theta$
$\quad (-1)du = \sin\theta\,d\theta$

$-8u + \frac{8}{3}u^3 + C$

$-8\cos\theta + \frac{8}{3}\cos^3\theta + C$

$-8\frac{\sqrt{4-x^2}}{2} + \frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C$

$\boxed{-4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + C}$

$\sin\theta = \dfrac{x}{2}$

$\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$
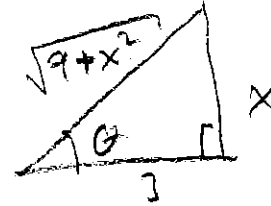
Trigonometric Substitution Method:
- A) Substitute, don't forget dx = ??dθ.
  Simplify (eliminate root)
- B) Use 7.2 methods for trig integrals.
- C) Draw a triangle and return to x.

$\tan\theta = x/3$

$2. \int \sqrt{9+x^2}\,dx$

$\boxed{x = 3\tan\theta}$

$dx = 3\sec^2\theta\,d\theta$

$\sec\theta = \dfrac{\sqrt{9+x^2}}{3}$

$\int \underbrace{\sqrt{9+9\tan^2\theta}}_{3\sec\theta}\, 3\sec^2\theta\,d\theta$

$+9\int \sec^3\theta\,d\theta$

$\dfrac{9}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + c$

$\dfrac{9}{2}\left(\dfrac{\sqrt{9+x^2}}{3}\cdot\dfrac{x}{3} + \ln\left|\dfrac{\sqrt{9+x^2}}{3} + \dfrac{x}{3}\right|\right) + c$

$= \dfrac{1}{2}x\sqrt{9+x^2} + \dfrac{9}{2}\left(\ln|\sqrt{9+x^2} + x| - \ln(3)\right) + c$

$= \dfrac{1}{2}x\sqrt{9+x^2} + \dfrac{9}{2}\ln|\sqrt{9+x^2} + x| + D$

3. $\int \dfrac{\sqrt{x^2-16}}{x}\,dx$

$\boxed{x = 4\sec\theta}$

$dx = 4\sec\theta\tan\theta\,d\theta$

$\sqrt{16(\sec^2\theta - 1)}$
$= \sqrt{16\tan^2\theta}$
$= 4\tan\theta$
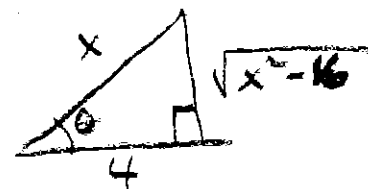
$\int \dfrac{\sqrt{\cancel{\sec^2\theta - 16}}}{4\sec\theta}\,\cancel{4\sec\theta\tan\theta\,d\theta}$

$\int 4\tan\theta \cdot \tan\theta\,d\theta$

$= 4\int \tan^2\theta\,d\theta$

$\sec\theta = x/4$

$= 4\int \sec^2\theta - 1\,d\theta$

$\theta = \sec^{-1}(x/4)$

$= 4(\tan\theta - \theta) + C$

$= 4\tan\theta - 4\theta + C$

$= 4\dfrac{\sqrt{x^2-16}}{4} - 4\sec^{-1}(x/4) + C = \boxed{\sqrt{x^2-16} - 4\sec^{-1}(x/4) + C}$

Completing the Square:

$$\sqrt{ax^2 + bx + c}$$

If you encounter a "**middle term**" (like **bx** above), then complete the square.

Example: $\sqrt{64 - 24x - 4x^2}$

i) Factor out the "a".

$$\sqrt{4(16 - 6x - x^2)} = 2\sqrt{16 - 6x - x^2}$$

ii) Add/subtract "half-middle squared"

Half of middle = (-6)/2 = -3

Squared = (-3)² = 9

$$2\sqrt{16 + 9 - 9 - 6x - x^2}$$

iii) Factor the perfect square

$$2\sqrt{25 - (x + 3)^2}$$

iv) Check your work!!!!!

Example:

$$\int \frac{x}{\sqrt{64 - 24x - 4x^2}}\,dx$$

$$= \int \frac{x}{2\sqrt{25 - (x+3)^2}}\,dx$$

$$\boxed{x+3 = 5\sin\theta} \rightarrow x = 5\sin\theta - 3 \qquad \sin\theta = \frac{x+3}{5}$$

$$dx = 5\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{(5\sin\theta - 3)}{\sqrt{25 - 25\sin^2\theta}}\, 5\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{(5\sin\theta - 3)}{5\cos\theta}\, 5\cos\theta\, d\theta$$

$$= \frac{1}{2}(-5\cos\theta - 3\theta) + C$$

$$= -\frac{5}{2}\cos\theta - \frac{3}{2}\theta + C$$

$$= -\frac{5}{2}\frac{\sqrt{25-(x+3)^2}}{5} - \frac{3}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + C$$